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Hyper-Reflexivity and the Factorization of Linear Functionals

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It is shown that hyper-reflexivity of a space of linear operators on a Hilbert space follows from a factorization property of linear functionals continuous in the weak operator topology. This provides new examples of hyper-reflexive algebras and new proofs for the hyper-reflexivity of the noncommutative disk algebras. © 1998 Academic Press

1. INTRODUCTION

Consider a complex Hilbert space $\mathfrak{H}$, and the algebra $\mathfrak{L}(\mathfrak{H})$ of bounded linear operators on $\mathfrak{H}$. A subspace $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ is said to be *reflexive* if every operator $T \in \mathfrak{L}(\mathfrak{H})$, with the property that $Tx \in [\mathfrak{M}x]^-$ for all $x \in \mathfrak{H}$, necessarily belongs to $\mathfrak{M}$. This property can be formulated in a different way. Given vectors $x, y \in \mathfrak{H}$, denote by $x \otimes y$ the functional defined on $\mathfrak{L}(\mathfrak{H})$ by

$$\langle T, x \otimes y \rangle = (Tx, y), \quad T \in \mathfrak{L}(\mathfrak{H}).$$

We will also denote by $[x \otimes y]_{\mathfrak{M}}$ the restriction of $x \otimes y$ to the subspace $\mathfrak{M}$. Then $\mathfrak{M}$ is reflexive if, for every $T \notin \mathfrak{M}$, there exist $x, y \in \mathfrak{H}$ satisfying $[x \otimes y]_{\mathfrak{M}} = 0$ and $(Tx, y) \neq 0$. A different formulation yet is given in terms of seminorms. Denote $d_{\mathfrak{M}}(T) = \inf\{ \|T - X\| : X \in \mathfrak{M}\}$ and $r_{\mathfrak{M}}(T) = \sup\{ |(Tx, y)| : x, y \in \mathfrak{H}, [x \otimes y]_{\mathfrak{M}} = 0, \|x\|, \|y\| \leqslant 1\}$. Then $\mathfrak{M}$ is reflexive if the equality $d_{\mathfrak{M}}(T) = 0$ is equivalent to $r_{\mathfrak{M}}(T) = 0$. Observe that we always have $r_{\mathfrak{M}} \leqslant d_{\mathfrak{M}}$. The linear space $\mathfrak{M}$ is said to be *hyper-reflexive*, or to satisfy a *distance formula* if there is a constant $C > 0$ such that $d_{\mathfrak{M}}(T) \leqslant C r_{\mathfrak{M}}(T)$ for every $T \in \mathfrak{L}(\mathfrak{H})$. The smallest constant C is called the *hyper-reflexivity* constant of $\mathfrak{M}$. These notions are usually introduced in terms of invariant subspaces when $\mathfrak{M}$ is an algebra. The above notion of reflexivity for linear spaces was introduced in [12]. Distance formulas were first proved in [2] for nest algebras.

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It has been known for some time that reflexivity of a subspace is related to, and sometimes can be deduced from, factorization properties of linear functionals on that subspace; see [3–5, 9–12] for results of this nature. Our purpose in this paper will be to show that hyper-reflexivity can be deduced from sufficiently strong factorization properties. As a consequence, the weakly closed algebra generated by an operator in the class $\mathbb{A}_{\aleph_0}$ [5] is hyper-reflexive with constant at most 3. Examples of such operators are contractions with dominating essential spectrum [13] and many weighted shifts, including the Bergman shift. In Section 4 we show that the hypothesis of our hyper-reflexivity result are satisfied if the commutant $\mathfrak{M}'$ of the space $\mathfrak{M}$ contains two isometries with orthogonal ranges. As a consequence, such subspaces are hyper-reflexive with constant at most 3. This situation covers the noncommutative Toeplitz algebra $\mathfrak{L}_n$ studied in [1] and [7] if $n \geqslant 2$.

The hyper-reflexivity of $\mathfrak{L}_n$ was first proved in [6] for $n=1$, and in [7] for $n \geqslant 2$. The reflexivity of $\mathfrak{L}_n$ was first proved in [14] for $n=1$, and in [1] for $n \geqslant 2$. In a related recent result, it is shown in [9] that the direct sum of two operators in $\mathbb{A}_{\aleph_0}$ is hyper-reflexive.

The remainder of the paper is organized as follows. Section 2 contains preliminaries about the factorization of linear functionals. In Section 3 we prove the main result about hyper-reflexivity.

The work in this paper was inspired by Ken Davidson's talk on the algebras $\mathfrak{L}_n$ given in the Wabash seminar, and by subsequent conversations with Bebe Prunaru.

2. PRELIMINARIES

Recall that the weak operator topology on $\mathfrak{L}(\mathfrak{H})$ is the weak topology generated by finite sums of functionals of the form $x \otimes y$, with $x, y \in \mathfrak{H}$. Reflexive subspaces are always closed in the weak operator topology. Therefore, for the rest of this section we fix a subspace $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$ which is closed in the weak operator topology. The following result is an easy consequence of the fact that $\mathfrak{L}(\mathfrak{H})$ can be viewed as the dual of the space of weak operator continuous functionals, and of the fact that the norm on weak operator continuous functionals is the projective tensor product norm (see Section 1.3 in [8]).

2.1 Lemma. *For every $T \in \mathfrak{L}(\mathfrak{H})$ we have*

$$d_{\mathfrak{M}}(T) = \sup \left\{ \left| \sum_{j=1}^{n} (Tx_j, y_j) \right| : \sum_{j=1}^{n} [x_j \otimes y_j]_{\mathfrak{M}} = 0, \sum_{j=1}^{n} \|x_j\| \, \|y_j\| \leqslant 1 \right\}.$$

We now proceed to describe the factorization properties relevant for this work. Given a number $\theta > 0$, we introduce the set $\mathscr{X}_\theta(\mathfrak{M})$ consisting of all norm continuous functionals φ on $\mathfrak{M}$ with the following property: given a finite number $h_1, h_2, ..., h_p$ of vectors in $\mathfrak{H}$, and a positive number ε, there exist $x, y \in \mathfrak{H}$ such that

(a) $\|x\|, \|y\| \leqslant 1$;

(b) $\|\varphi - [x \otimes y]_{\mathfrak{M}}\| < \theta + \varepsilon$; and

(c) $\|[x \otimes h_j]_{\mathfrak{M}}\| + \|[h_j \otimes y]_{\mathfrak{M}}\| < \varepsilon$ for $j = 1, 2, ..., p$.

We claim that the vectors x and y can actually be chosen so that the following additional condition is satisfied:

(d) $|(x, h_j)| + |(h_j, y)| < \varepsilon$ for $i = 1, 2, ..., p$.

To do this, observe that the vector x in this definition can be replaced by its projection on the closed subspace $\mathfrak{R}_*$ generated by $\{T^*\xi : T \in \mathfrak{M}, \xi \in \mathfrak{H}\}$. Indeed, this replacement does not change any of the tensor products, and it can only decrease the norm of x. The vectors h_j can now be written as $h_j = \sum_{i=1}^N T_i^* \xi_{ij} + u_j + v_j$, with $T_i \in \mathfrak{M}$, $\|u_j\| < \varepsilon/4$, and $v_j \perp \mathfrak{R}_*$. Thus we have

$$|(x, h_j)| < \sum_{i=1}^{N} |(x, T_i^* \xi_{ij})| + \varepsilon/4 \leqslant \sum_{i=1}^{N} \|[x \otimes \xi_{ij}]_{\mathfrak{M}}\| \, \|T_i\| + \varepsilon/4 < \varepsilon/2,$$

provided that x is chosen so that $\|[x \otimes \xi_{ij}]_{\mathfrak{M}}\|$ is sufficiently small. Similar considerations about the vector y show that condition (d) can indeed be satisfied.

With these observations, it can be proved like in [5] that the set $\mathscr{X}_\theta(\mathfrak{M})$ is norm closed, convex, and balanced. If $\gamma > \theta$, we say that $\mathfrak{M}$ has property $X_{\theta, \gamma}$ if the set $\mathscr{X}_\theta(\mathfrak{M})$ contains every weak operator continuous functional of norm $\leqslant \gamma$ on $\mathfrak{M}$.

The reader familiar with [5] will note that this definition is somewhat at variance with the usual one. There are two reasons for the difference. The first is that our definition works in arbitrary Hilbert spaces, without separability. The second is that the version of condition (c) used in [5] is unnecessarily restrictive. The arguments from [5] can be adapted to our defintion with little or no change. Thus we have the following result (Cf. Theorem 2.11 in [5]).

2.2 THEOREM. *If $\mathfrak{M}$ has property $X_{\theta, \gamma}$ then it also has property $X_{0, \gamma - \theta}$. More precisely, if φ is a weak operator continuous functional on $\mathfrak{M}$, ε is a positive number, and $h_1, h_2, ..., h_p$ is a finite sequence of vectors in $\mathfrak{H}$, there exist vectors $x, y \in \mathfrak{H}$ such that*

(a) $\|x\|, \|y\| \leqslant (\gamma-\theta)^{-1/2}\|\varphi\|^{1/2}+\varepsilon$;
(b) $\varphi=[x\otimes y]_{\mathfrak{M}}$;
(c) $\|[x\otimes h_j]_{\mathfrak{M}}\|+\|[h_j\otimes y]_{\mathfrak{M}}\|<\varepsilon$ *for* $j=1, 2, ..., p$; *and*
(d) $|(x, h_j)|+|(h_j, y)|<\varepsilon$ *for* $j=1, 2, ..., p$.

Stronger factorization theorems can in fact be proved for arrays of weak operator continuous functionals on $\mathfrak{M}$. Fix a natural number n. As in [3], we will say that $\mathfrak{M}$ has *property* $(\mathbb{A}_n^{\sim})$ if for every $\varepsilon>0$ there exists $\delta=\delta(n, \varepsilon)$ with the following property. Given weak operator continuous functionals $\{\varphi_{ij}: 1\leqslant i, j\leqslant n\}$ on $\mathfrak{M}$, and vectors $\{x_i, y_i: 1\leqslant i\leqslant n\}\subset\mathfrak{H}$ satisfying the inequalities

$$\|\varphi_{ij}-[x_i\otimes y_j]_{\mathfrak{M}}\|<\delta, \qquad 1\leqslant i, \quad j\leqslant n,$$

there exist vectors $\{x_i', y_i': \leqslant i\leqslant n\}\subset\mathfrak{H}$ such that

$$\varphi_{ij}=[x_i'\otimes y_j']_{\mathfrak{M}}, \qquad 1\leqslant i, \quad j\leqslant n,$$

and

$$\|x_i'-x_i\|, \|y_i'-y_i\|<\varepsilon, \qquad 1\leqslant i\leqslant n.$$

The following result is from [3].

2.3 THEOREM. *If a subspace* $\mathfrak{M}$ *has property* $X_{\theta,\gamma}$ *then it also has property* $(\mathbb{A}_n^{\sim})$ *for every natural number* n.

Property $(\mathbb{A}_n^{\sim})$ can be formulated equivalently with ultraweakly continuous functionals in place of weak operator continuous ones. Note that when property $X_{\theta,\gamma}$ is satisfied, all ultraweakly continuous functionals on $\mathfrak{M}$ are in fact weak operator continuous on $\mathfrak{M}$ as well.

3. HYPER-REFLEXIVITY

3.1. THEOREM. *Let* $\mathfrak{M}$ *be a subspace of* $\mathfrak{L}(\mathfrak{H})$ *which has property* $X_{\theta,\gamma}$ *for some* $\gamma>\theta>0$. *If* $\mathfrak{M}$ *is closed in the weak operator topology then* $\mathfrak{M}$ *is hyper-reflexive with constant at most* $1+2/(\gamma-\theta)$.

Proof. Fix an operator $T\in\mathfrak{L}(\mathfrak{H})$ and vectors $\{x_i, y_i: 1\leqslant i\leqslant n\}\subset\mathfrak{H}$ such that $\sum_{i=1}^n [x_i\otimes y_i]_{\mathfrak{M}}=0$ and $\sum_{i=1}^n \|x_i\|\,\|y_i\|\leqslant 1$. By Lemma 2.1, the theorem will be proved if we show that

$$\left|\sum_{i=1}^n (Tx_i, y_i)\right|\leqslant Cr_{\mathfrak{M}}(T), \tag{3.2}$$

with $C = 1 + 2/(\gamma - \theta)$. We may, and shall, assume that $\|x_i\| = \|y_i\|$ for all i. Let ε be a positive number, and let $\delta = \delta(2n, \varepsilon)$ be the constant given by the fact that $\mathfrak{M}$ has property $(\mathbb{A}_{\widetilde{2n}})$. By Theorem 2.2 it is possible to find vectors $\{u_i, v_i\colon 1 \leqslant i \leqslant n\} \subset \mathfrak{H}$ with the following properties:

$$\|u_i\|, \|v_i\| \leqslant (\gamma - \theta)^{-1/2} \|x_i\| + \varepsilon, \qquad 1 \leqslant i \leqslant n, \tag{3.3}$$

$$[u_i \otimes v_i]_{\mathfrak{M}} = [x_i \otimes y_i]_{\mathfrak{M}} \qquad 1 \leqslant i \leqslant n, \tag{3.4}$$

$$\|[u_i \otimes y_i]_{\mathfrak{M}}\|, \|[x_i \otimes v_j]_{\mathfrak{M}}\| < \delta, \qquad 1 \leqslant i, j \leqslant n, \tag{3.5}$$

and

$$|(Tx_i, v_j)|, |(Tu_i, y_j)| < \varepsilon, \qquad 1 \leqslant i, j \leqslant n. \tag{3.6}$$

By choosing the vectors u_i, v_i inductively we can also guarantee that

$$\|u_i \otimes v_j\| < \delta \qquad \text{for} \quad i \neq j, \tag{3.7}$$

and

$$|(u_i, v_j)|, |(Tu_i, v_j)| < \varepsilon \qquad \text{for} \quad i \neq j. \tag{3.8}$$

Property $(\mathbb{A}_{\widetilde{2n}})$ with u_i and v_i in place of x_{n+i} and y_{n+i}, respectively, yields vectors x'_i, y'_i, u'_i and v'_i, $1 \leqslant i \leqslant n$, such that

$$\|x'_i - x_i\|, \|y'_i - y_i\|, \|u'_i - u_i\|, \|v'_i - v_i\| < \varepsilon, \quad 1 \leqslant i \leqslant n, \tag{3.9}$$

$$[x'_i \otimes y'_j]_{\mathfrak{M}} = [x_i \otimes y_j]_{\mathfrak{M}}, \qquad 1 \leqslant i, j \leqslant n, \tag{3.10}$$

$$[x'_i \otimes v'_j]_{\mathfrak{M}} = [u'_i \otimes y'_j]_{\mathfrak{M}} = 0 \qquad 1 \leqslant i, j \leqslant n, \tag{3.11}$$

$$[u'_i \otimes v'_i]_{\mathfrak{M}} = [x_i \otimes y_i]_{\mathfrak{M}}, \qquad 1 \leqslant i \leqslant n, \tag{3.12}$$

and

$$[u'_i \otimes v'_j]_{\mathfrak{M}} = 0 \qquad \text{for} \quad i \neq j. \tag{3.13}$$

Inequalities (3.6) and (3.8) combined with (3.9) easily yield

$$|(Tx'_i, v'_j)| = O(\varepsilon), |(Tu'_i, y'_j)| = O(\varepsilon), \qquad 1 \leqslant i, j \leqslant n, \tag{3.14}$$

and

$$|(u'_i, v'_j)| = O(\varepsilon), |(Tu'_i, v'_j)| = O(\varepsilon) \qquad \text{for} \quad i \neq j, \tag{3.15}$$

where $O(\varepsilon)$ denotes a quantity bounded by a constant times ε, and the constant can be determined entirely from the initial data T, n, x_i, y_i, θ and γ. Inequalities (3.9) also imply that

$$\begin{aligned}|(Tx_i, y_i)-(Tx'_i, y'_i)| &\leqslant \|Tx_i\|\ \|y_i-y'_i\|+\|T\|\ \|x_i-x'_i\|\ \|y'_i\|\\ &\leqslant \|Tx_i\|\,\varepsilon+\|T\|\ \varepsilon(\|y_i\|+\varepsilon),\end{aligned}$$

and therefore

$$\left|\sum_{i=1}^{n}(Tx_i, y_i)\right|\leqslant\left|\sum_{i=1}^{n}(Tx'_i, y'_i)\right|+O(\varepsilon). \tag{3.16}$$

In order to estimate the right hand side of (3.16) observe that (3.10), (3.11) and (3.12) imply $[(x'_i-u'_i)\otimes(y'_i+v'_i)]_{\mathfrak{M}}=0$, so that

$$|(T(x'_i-u'_i), y'_i+v'_i)|\leqslant r_{\mathfrak{M}}(T)\ \|x'_i-u'_i\|\ \|y'_i+v'_i\|.$$

Expanding the scalar product in the left hand side yields

$$\begin{aligned}|(Tx'_i, y'_i)-(Tu'_i, v'_i)| &\leqslant r_{\mathfrak{M}}(T)\ \|x'_i-u'_i\|\ \|y'_i+v'_i\|+|(Tx'_i, v'_i)|+|(Tu'_i, y'_i)|\\ &\leqslant r_{\mathfrak{M}}(T)\ \|x'_i-u'_i\|\ \|y'_i+v'_i\|+O(\varepsilon)\end{aligned} \tag{3.17}$$

by (3.14). Now, (3.15) implies that

$$\begin{aligned}\|x'_i-u'_i\|^2 &= \|x'_i\|^2+\|u'_i\|^2+O(\varepsilon)\\ &\leqslant(\|x_i\|+\varepsilon)^2+(\|u_i\|+\varepsilon)^2+O(\varepsilon)\\ &\leqslant(1+(\gamma-\theta)^{-1})\ \|x_i\|^2+O(\varepsilon)\end{aligned}$$

and, analogously,

$$\|y'_i+v'_i\|^2\leqslant(1+(\gamma-\theta)^{-1})\ \|x_i\|^2+O(\varepsilon).$$

These estimates combined with (3.16) and (3.17) yield

$$\begin{aligned}\left|\sum_{i=1}^{n}(Tx_i, y_i)\right| &\leqslant\left|\sum_{i=1}^{n}(Tu'_i, v'_i)\right|+r_{\mathfrak{M}}(T)(1+(\gamma-\theta)^{-1})\sum_{i=1}^{n}\|x_i\|^2+O(\varepsilon)\\ &\leqslant\left|\sum_{i=1}^{n}(Tu'_i, v'_i)\right|+(1+(\gamma-\theta)^{-1})\ r_{\mathfrak{M}}(T)+O(\varepsilon).\end{aligned} \tag{3.18}$$

Finally, observe that

$$\left[\sum_{i=1}^{n}u'_i\otimes\sum_{i=1}^{n}v'_i\right]_{\mathfrak{M}}=\sum_{i=1}^{n}[u'_i\otimes v'_i]_{\mathfrak{M}}+\sum_{j\neq i}[u'_i\otimes v'_j]_{\mathfrak{M}}=\sum_{i=1}^{n}[x_i\otimes y_i]_{\mathfrak{M}}=0,$$

and hence

$$\left|\left(T\sum_{i=1}^{n} u_i', \sum_{i=1}^{n} v_i'\right)\right| \leqslant r_{\mathfrak{M}}(T) \left\|\sum_{i-1}^{n} u_i'\right\| \left\|\sum_{i=1}^{n} v_i'\right\|.$$

Thus

$$\begin{aligned}\left|\sum_{i=1}^{n} (Tu_i', v_i')\right| &\leqslant r_{\mathfrak{M}}(T) \left\|\sum_{i=1}^{n} u_i'\right\| \left\|\sum_{i=1}^{n} v_i'\right\| + \sum_{i\neq j} |(Tu_i', v_j')| \\ &\leqslant r_{\mathfrak{M}}(T) \left\|\sum_{i=1}^{n} u_i'\right\| \left\|\sum_{i=1}^{n} v_i'\right\| + O(\varepsilon).\end{aligned}$$

By (3.15) and (3.9)

$$\begin{aligned}\left\|\sum_{i=1}^{n} u_i'\right\|^2 &= \sum_{i=1}^{N} \|u_i'\|^2 + O(\varepsilon) = \sum_{i=1}^{n} \|u_i\|^2 + O(\varepsilon) \\ &\leqslant (\gamma-\theta)^{-1} \sum_{i=1}^{n} \|x_i\|^2 + O(\varepsilon) \leqslant (\gamma-\theta)^{-1} + O(\varepsilon)\end{aligned}$$

and, similarly,

$$\left\|\sum_{i=1}^{n} v_i'\right\|^2 \leqslant (\gamma-\theta)^{-1} + O(\varepsilon).$$

Therefore

$$\left|\sum_{i=1}^{n} (Tu_i', v_i')\right| \leqslant (\gamma-\theta)^{-1}\, r_{\mathfrak{M}}(T) + O(\varepsilon),$$

so that

$$\left|\sum_{i=1}^{n} (Tx_i, y_i)\right| \leqslant (1 + 2/(\gamma-\theta)\, r_{\mathfrak{M}}(T) + O(\varepsilon)$$

by (3.18). Inequality (3.2) is now obtained by letting ε tend to zero. ∎

4. COMMUTANTS OF ISOMETRIES WITH ORTHOGONAL RANGES

Throughout this section we fix two isometries U, V in $\mathfrak{L}(\mathfrak{H})$ such that $U^*V=0$, and a subspace $\mathfrak{M}\subset\mathfrak{L}(\mathfrak{H})$ which commutes with both U and V. The subspace $\mathfrak{M}$ will be assumed to be closed in the weak operator topology.

4.1 LEMMA. *Given a finite set* $\{h_1, h_2, ..., h_p\} \subset \mathfrak{H}$ *and* $\varepsilon > 0$, *there exists an isometry* W *in* $\mathfrak{M}'$ *such that*

(a) $W\mathfrak{H} \subset U\mathfrak{H}$; *and*

(b) $\|W^* h_j\| < \varepsilon$ *for* $j = 1, 2, ..., p$.

Proof. The operator W will be constructed as a product $W = W_n = T_1 T_2 \cdots T_n$ with $T_j \in U, V$ and $T_1 = U$. The basic observation is that, since U and V have orthogonal ranges,

$$\sum_{j=1}^{p} \|(W_n U)^* h_j\|^2 + \sum_{j=1}^{p} \|(W_n V)^* h_j\|^2 \leqslant \sum_{j=1}^{p} \|W_n^* h_j\|^2.$$

Then T_{n+1} can be chosen so that

$$\sum_{j=1}^{p} \|W_{n+1}^* h_j\|^2 \leqslant \tfrac{1}{2} \sum_{j=1}^{p} \|W_n^* h_j\|^2.$$

Clearly the conclusions of the lemma are satisfied for large n. ∎

We will need a well-known fact about Fourier series. Denote by H^∞ the Banach algebra of bounded analytic functions on the unit disk $\{\lambda \in \mathbb{C}: |\lambda| < 1\}$. A function $u \in H^\infty$ can be written as a power series $u(\lambda) = \sum_{j=0}^{\infty} u_n \lambda^n$, $|\lambda| < 1$, and for every natural number n there is a smallest constant C_n, such that

$$|u_1 + u_2 + \cdots + u_n| \leqslant C_n \|u\|_\infty$$

for every $u \in H^\infty$. The constant C_n is no larger than the norm of the function $\sum_{k=1}^{n} e^{2\pi i k t}$ in $L^1(0, 1)$, and a calculation analogous to the estimate of the Lebesgue constants (cf. Section II.12 in [15]) shows that

$$C_n = O(\log n) \qquad \text{as} \quad n \to \infty. \tag{4.2}$$

4.3. THEOREM. *A weak operator closed subspace* $\mathfrak{M} \subset \mathfrak{L}(\mathfrak{H})$, *whose commutant contains two isometries with orthogonal ranges, has property* $X_{0,1}$.

4.4. COROLLARY. *Under the hypothesis of Theorem* 4.3, $\mathfrak{M}$ *is hyper-reflexive with constant at most* 3.

Proof. Since the set $\mathscr{X}_0$ is closed, convex, and balanced, it will suffice to show that $[u \otimes v]_{\mathfrak{M}}$ belongs to $\mathscr{X}_0$ whenever $u, v \in \mathfrak{H}$ and $\|u\|, \|v\| \leqslant 1$. Fix therefore such vectors u and v, a finite family $h_1, h_2, ..., h_p \subset \mathfrak{H}$, and $\varepsilon > 0$. Choose first isometries V and W in the commutant of $\mathfrak{M}$ such that

$W^*V=0$ and $\|W^*h_j\|<\varepsilon/3$ for $j=1, 2, ..., p$. This is possible by Lemma 4.1. For each positive integer n let us set

$$x_n=n^{-1/2}\sum_{k=1}^{n} W^kVu, \qquad y_n=n^{-1/2}\sum_{k=1}^{n} W^kVv.$$

If $T\in\mathfrak{M}$ we have

$$\begin{aligned}(Tx_n, y_n) &= \frac{1}{n}\sum_{k,\ell=1}^{n}(TW^kVu, W^lVv)=\frac{1}{n}\sum_{k=1}^{n}(W^kVTu, W^kVu)\\ &\quad +\frac{1}{n}\sum_{k>\ell}(W^kVTu, W^lVv)+\frac{1}{n}\sum_{k<\ell}(W^kVTu, W^lVv)\\ &=(Tu, v)+\frac{1}{n}\sum_{k>\ell}(W^{k-\ell}VTu, Vv)+\sum_{k<\ell}(VTu, W^{l-k}Vv)\\ &=(Tu, v)\end{aligned}$$

since V and W have orthogonal ranges. Thus

$$[x_n\otimes y_n]_{\mathfrak{M}}=[u\otimes v]_{\mathfrak{M}} \qquad n=1, 2, ...$$

A similar calculation shows that $\|x_n\|=\|u\|\leqslant 1$ and $\|y_n\|=\|v\|\leqslant 1$. To conclude that $[u\otimes v]_{\mathfrak{M}}\in\mathscr{X}_0$ it will suffice to show that $\|[h_j\otimes y_n]_{\mathfrak{M}}\|<\varepsilon/2$ and $\|[x_n\otimes h_j]_{\mathfrak{M}}\|<\varepsilon/2$, $j=1, 2, ..., p$, provided that n is large enough. Let $T\in\mathfrak{M}$ be an operator with $\|T\|\leqslant 1$. Since $x_n\in Wh$ we also have $Tx_n\in Wh$, and therefore

$$\begin{aligned}\langle T, [x_n\otimes h_j]_{\mathfrak{M}}\rangle &=(Tx_n, h_j)=(WW^*Tx_n, h_j)=(W^*Tx_n, W^*h_j)\\ &\leqslant\|T\|\,\|x_n\|\,\|W^*h_j\|<\varepsilon/3\end{aligned}$$

by the choice of W. Thus $\|[x_n\otimes h_j]_{\mathfrak{M}}\|<\varepsilon/3$. In order to estimate $\|[h_j\otimes y_n]_{\mathfrak{M}}\|$ we write $h_j=a_j+b_j$, with $a_j\in\ker W^*$ and $\|b_j\|<\varepsilon/3$, whence

$$\|[h_j\otimes y_n]_{\mathfrak{M}}\|\leqslant\|[a_j\otimes y_n]_{\mathfrak{M}}\|+\varepsilon/3, \qquad j=1, 2, ..., p.$$

It will then suffice to show that $\|[a\otimes y_n]_{\mathfrak{M}}| \to 0$ as $n\to\infty$ for every $a\in\ker W^*$. To do this, fix again an operator $T\in\mathfrak{M}$. We claim that the function $u(\lambda)=\sum_{k=0}^{\infty}\lambda^k(Ta, W^kVv)$ belongs to H^∞ and

$$\|u\|_\infty\leqslant\|T\|\,\|a\|\,\|v\|.$$

To verify this claim consider the vectors

$$e_\lambda = \sum_{k=0}^{\infty} \bar{\lambda}^k W^k a \qquad \text{and} \qquad f_\lambda = \sum_{k=0}^{\infty} \bar{\lambda}^k W^k V v, \qquad |\lambda| < 1.$$

Since a and Vv belong to the kernel of W^*, the terms in the series defining e_λ and f_λ are pairwise orthogonal, whence

$$\|e_\lambda\|^2 = (1 - |\lambda|^2)^{-1} \|a\| \qquad \text{and} \qquad \|f_\lambda\|^2 = (1 - |\lambda|^2)^{-1} \|v\|, \qquad |\lambda| < 1.$$

Now observe that

$$(Te_\lambda, f_\lambda) = \sum_{k,\,\ell=1}^{\infty} \lambda^{-k} \lambda^l (W^k Ta, W^l Vv),$$

and

$$(W^k Ta, W^l Vv) = (W^{k-l} Ta, Vv) = 0$$

if $k > \ell$. Thus

$$\begin{aligned}
(Te_\lambda, f_\lambda) &= \sum_{k \leqslant \ell} \bar{\lambda}^k \lambda^\ell (W^k Ta, W^\ell Vv) = \sum_{k,\, m=0}^{\infty} \bar{\lambda}^k \lambda^{k+m} (W^k Ta, W^{k+m} Vv) \\
&= \sum_{k,\, m=0}^{\infty} \bar{\lambda}^k \lambda^{k+m} (Ta, W^m Vv) \\
&= \left(\sum_{k=0}^{\infty} \bar{\lambda}^k \lambda^k \right) \left(\sum_{m=0}^{\infty} \lambda^m (Ta, W^m Vv) \right) = (1 - |\lambda|^2)^{-1} u(\lambda),
\end{aligned}$$

from which we conclude that

$$|u(\lambda)| = (1 - |\lambda|^2)\, |(Te_\lambda, f_\lambda)| \leqslant (1 - |\lambda|^2)\, \|T\|\, \|e_\lambda\|\, \|f_\lambda\| = \|T\|\, \|a\|\, \|v\|,$$

as claimed. Therefore

$$\begin{aligned}
|\langle T, [a \otimes y_n]_{\mathfrak{M}} \rangle| = |(Ta, y_n)| &= n^{-1/2} \sum_{k=1}^{n} (Ta, W^k Vv)| \\
&\leqslant n^{-1/2} C_n \|u\|_\infty \leqslant n^{-1/2} C_n \|T\|\, \|a\|\, \|v\|,
\end{aligned}$$

and hence $\|[a \otimes y_n]_{\mathfrak{M}}\| \leqslant n^{-1/2} C_n \|a\|\, \|v\| \to 0$ as $n \to \infty$ by (4.2). ∎

The reader will recognize in the above proof an analytic Toeplitz operator, and the evaluation of its symbol using the Szegö kernel. It was expedient to write down the explicit calculations. It may be amusing to ponder why the choices $x_n = W^n Vu$, $y_n = W^n Vv$ or $x_n = n^{-1/2} \sum_{k=1}^{n} W^k u$, $y_n = n^{-1/2} \sum_{k=1}^{n} W^k v$ will not work in the above proof.

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